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CONSEQUENCES OF UNIFORM INTEGRABILITY FOR SIMULATION

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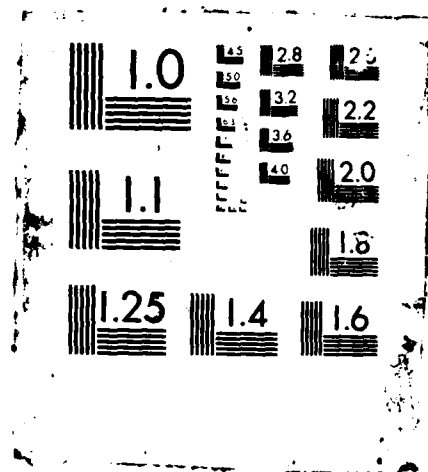
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## CONSEQUENCES OF UNIFORM INTEGRABILITY FOR SIMULATION

by

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## 1. INTRODUCTION

Let  $\mathbf{X} = \{X(t) : t \geq 0\}$  be a real-valued stochastic process representing the output of a simulation. (To incorporate discrete-time sequences  $\{X_n : n \geq 0\}$  into our framework, we set  $X(t) = X_{[t]}$ , where  $[t]$  is the greatest integer less than or equal to  $t$ .) Suppose that  $\mathbf{X}$  is ergodic, in the sense that there exists  $\mu \in \mathbb{R}$  such that

$$(1.1) \quad \bar{X}(t) \equiv \frac{1}{t} \int_0^t X(s) ds \Rightarrow \mu$$

as  $t \rightarrow \infty$ . Considerable attention has been focused, in the simulation literature, on the problem of producing confidence intervals for the estimand  $\mu$ ; this is known as the steady-state simulation problem. (See Chapter 8 of LAW and KELTON (1983) for details.)

Virtually all methods proposed for the steady-state simulation problem require that  $\mathbf{X}$  obey a central limit theorem; that is, there should exist a constant  $\sigma$  for which

$$(1.2) \quad t^{1/2}(\bar{X}(t) - \mu) \Rightarrow \mathcal{N}(0, \sigma^2)$$

as  $t \rightarrow \infty$ . Sufficient conditions for (1.2) are well-known. In particular, conditions for the validity of (1.2) are known when  $\mathbf{X}$  is regenerative (SMITH (1955)), strong mixing (HALL and HEYDE (1980), pp. 127-153),  $\phi$ -mixing (BILLINGSLEY (1968), pp. 166-193) and associated (NEWMAN and WRIGHT (1981)).

Recently, several contributions to the steady-state simulation problem have required a moment version of (1.2):

$$(1.3) \quad E\{[t^{1/2}(\bar{X}(t) - \mu)]^k\} \rightarrow \sigma^k E N(0,1)^k .$$

Of course, relation (1.3) is a statement about uniform integrability in the limit theorem obtained by taking the  $k^{\text{th}}$  power of both sides of (1.2). In this paper, it is our intention to establish (1.3) for the class of regenerative processes satisfying a relatively mild moment assumption, thereby showing that (1.3) is a condition which is in force for a large class of steady-state simulations. In particular, since general state space Markov chains can frequently be made regenerative (see ATHREYA and NEY (1978)), the uniform integrability result obtained here applies to a larger class of simulations than one would initially expect.

As indicated previously, (1.3) is a limit theorem which enjoys a number of simulation-related applications:

- (i) limit theory for the method of replications (see GLYNN (1986));
- (ii) efficiency of conditional Monte Carlo for semi-Markov processes (see FOX and GLYNN (1986));
- (iii) large-sample theory for indirect estimation via  $L = \lambda W$  (GLYNN and WHITT (1986));
- (iv) asymptotics for control variates as applied to steady-state simulation (see GLYNN and IGLEHART (1986a));
- (v) large-sample theory for confidence intervals (GLYNN and IGLEHART (1986b)).

In this paper, we shall also present several additional applications of uniform integrability to simulation. Both applications concern the constant  $\sigma$  appearing in (1.2); as is well known, estimation of  $\sigma$  is the central problem in producing steady-state confidence intervals for  $\mu$ .

Loosely speaking, the first application of uniform integrability involves showing that  $\sigma$  has two different representations, corresponding to whether  $X$  is viewed as a regenerative process or an (asymptotically) stationary process. The second topic concerns the asymptotic variance of the regenerative variance estimator for  $\sigma$ .

This paper is organized as follows. Section 2 is devoted to precise statements of the main results of this paper, while Sections 3 to 5 contain the proofs of the results discussed in Section 2.

## 2. STATEMENT OF MAIN RESULTS

Let  $X = \{X(t): t \geq 0\}$  be a (possibly) delayed real-valued regenerative process with regeneration times  $0 \leq T(0) < T(1) < \dots$ . For convenience, set  $T(-1) = 0$ , and let

$$Y_1 = \int_{T(i-1)}^{T(i)} X(s) ds ,$$

$$\tilde{Y}_1 = \int_{T(i-1)}^{T(i)} |X(s)| ds , \text{ and}$$

$$\tau_1 = T(1) - T(i-1) .$$

Then, if  $E(\tilde{Y}_1 + \tau_1) < \infty$ , it is well known (see, for example SMITH (1955)) that the strong law version of (1.1) holds, namely

$$\bar{X}(t) \rightarrow \mu \quad \text{a.s.}$$

as  $t \rightarrow \infty$ , where  $\mu = EY_1/E\tau_1$ . If, furthermore,  $E(\tilde{Y}_1^2 + \tau_1^2) < \infty$ , then the central limit theorem (1.2) is valid (see [18]):

$$(2.1) \quad t^{1/2}(\bar{X}(t) - \mu) \Rightarrow \sigma N(0,1)$$

where  $\sigma = (EZ_1^2/E\tau_1)^{1/2}$  and  $Z_1 = Y_1 - \mu\tau_1$ . Our first result gives conditions under which the left-hand side of (2.1) is  $p^{\text{th}}$  moment uniformly integrable.

(2.2) THEOREM. For  $p \geq 2$ , if  $E(\tilde{Y}_0^p + \tilde{Y}_1^p + \tau_0^p + \tau_1^p) < \infty$ , then  $\{t^{p/2}|\bar{X}(t) - \mu|^p : t > 0\}$  is uniformly integrable.

An immediate corollary to this result are conditions which allow one to interchange limit and expectation in (2.1).

(2.3) COROLLARY. For  $p \geq 2$ , if  $E(\tilde{Y}_0^p + \tilde{Y}_1^p + \tau_0^p + \tau_1^p) < \infty$ , then for  $0 \leq \alpha \leq p$ ,

$$(2.4) \quad t^{\alpha/2} E|\bar{X}(t) - \mu|^\alpha \rightarrow \sigma^\alpha E|N(0,1)|^\alpha$$

as  $t \rightarrow \infty$ .

For  $p=2$ , (2.4) has been studied previously by CHUNG (1966), p. 102 (the proof given there in the Markov chain case easily extends to the regenerative setting), and by SMITH (1955). Our result weakens the conditions given there somewhat, and extends the result to  $p > 2$ .

We now turn to describing the first of the two applications mentioned in Section 1. The limit theorem (2.1) shows that the constant  $\sigma$  may be computed via  $\sigma^2 = EZ_1^2/E\tau_1$ . An alternative representation for  $\sigma$  may be obtained by taking limits in (2.4) with  $p = 2$ :

$$(2.5) \quad tE(\bar{X}(t) - \mu)^2 \rightarrow \sigma^2$$



as  $t \rightarrow \infty$ . The left-hand side of (2.5) has a particularly convenient form if  $\mathbf{X}$  is a stationary regenerative process. If  $\mathbf{X}$  is stationary regenerative under  $\hat{P}$ , then  $(\hat{E}(\cdot))$  denotes expectation relative to  $\hat{P}$

$$(2.6) \quad t\hat{E}(X(t)-\mu)^2 = 2 \int_0^t \left(1 - \frac{s}{t}\right) \hat{E}((X(0)-\mu) \cdot (X(s)-\mu))ds$$

Letting  $t \rightarrow \infty$ , one would formally expect that

$$\sigma^2 = 2 \int_0^{\infty} \hat{E}((X(0)-\mu)(X(s)-\mu))ds .$$

Our first application concerns finding precise conditions under which

$$(2.7) \quad EZ_1^2/E\tau_1 = 2 \int_0^{\infty} \hat{E}((X(0)-\mu)(X(s)-\mu))ds$$

where  $\hat{P}$  is the stationary version of the regenerative process associated with  $P$ . (See HEYMAN and SOBEL (1982), pp. 374-379.)

(2.8) THEOREM. Suppose either that:

- (i)  $\tau_1$  has a Lebesgue density component in its distribution,  $\mathbf{X}$  is bounded, and  $E\tau_1^3 < \infty$ , or
- (ii)  $\tau_1$  has an aperiodic distribution concentrated on  $\{1, 2, \dots\}$  and  $E(\tilde{Y}_1^3 + \tau_1^3) < \infty$ .

Then, (2.7) is valid.

An important special case of Theorem 2.8 is obtained by specializing to Markov processes. Let  $\mathbf{Y} = \{Y(t): t \geq 0\}$  be an irreducible positive recurrent continuous-time Markov chain on state space  $S \subseteq \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ .

Given a bounded real-valued function  $f$  defined on  $S$ , set  $X(t) = f(Y(t))$ . For fixed  $i \in S$ , let  $T(0) = \inf\{s \geq 0: Y(s) = i\}$ ,  $T(n) = \inf\{s > T(n-1): Y(s-) \neq i, Y(s) = i\}$  for  $n \geq 1$ . The sequence  $\{T(n): n \geq 0\}$  constitutes regeneration times for  $X$ . Since  $\tau_1$  has a density, Theorem 2.8 (i) applies and we get the following corollary.

(2.9) COROLLARY. Suppose  $E_1 T(1)^3 < \infty$ . Then

$$(2.10) \quad E_1 \left( \int_0^{T(1)} (f(Y(s)) - \mu) ds \right)^2 / E_1 T(1) = 2 \int_0^\infty E_\pi (f(Y(0)) - \mu)(f(Y(s)) - \mu) ds$$

where  $\mu = E_\pi f(X(0))$  (the integral on the right-hand side of (2.10) converges absolutely).

Our second application concerns the regenerative estimator for the constant  $\sigma$  appearing in (2.1). It is well known that if  $E(\tilde{Y}_1^2 + \tau_1^2) < \infty$ , then

$$v(t) = \begin{cases} \frac{1}{t} \sum_{k=1}^{N(t)} (Y_k - \bar{X}(t)\tau_k)^2; & N(t) \geq 1 \\ 0 & ; N(t) \geq 0 \end{cases}$$

converges to  $\sigma^2 = EZ_1^2/E\tau_1$  a.s.; the estimator for  $\sigma$  is then given by  $s(t) = v(t)^{1/2}$ . To study coverage problems associated with regenerative confidence intervals, it is of interest to study the correlation between the point estimate  $\bar{X}(t)$  and the standard deviation estimate  $s(t)$  (see, for example, BRATLEY, FOX, and SCHRAGE (1983), p. 113). A further interesting point involves the variability of  $s(t)$  as a function of the

regeneration state chosen. Both these questions require evaluating the asymptotic moment structure connecting  $\bar{X}(t)$  and  $s(t)$ .

To study the asymptotic moment structure, we use a joint limit theorem for  $\bar{X}(t)$  and  $s(t)$ ; if  $E(\tilde{Y}_1^4 + \tau_1^4) < \infty$ , then

$$(2.11) \quad t^{1/2}(\bar{X}(t)-\mu, s(t)-\sigma) \Rightarrow N(0, C)$$

where  $N(0, C)$  is a bivariate normal r.v. with mean vector 0 and covariance matrix  $C$  given by

$$C = \frac{1}{E\tau_1} \begin{pmatrix} EZ_1^2 & (2\sigma)^{-1}(EA_1Z_1 - \lambda EZ_1^2) \\ (2\sigma)^{-1}(EA_1Z_1 - \lambda EZ_1^2) & (4\sigma^2)^{-1}(EA_1^2 - 2\lambda EA_1Z_1 + \lambda^2 EZ_1^2) \end{pmatrix}$$

where  $A_1 = Z_1^2 - \sigma^2\tau_1$ ,  $\lambda = 2EZ_1\tau_1/E\tau_1$ . (See GLYNN and IGLEHART (1986c).)

The continuous mapping principle, as applied to (2.11), yields

$$(2.12) \quad t(s(t)-\sigma)^2 \rightarrow C_{22}^{1/2}N(0,1)$$

and

$$(2.13) \quad t(\bar{X}(t)-\mu)(s(t)-\sigma) \Rightarrow k(N(0, C))$$

when  $k: \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined by  $k(x_1, x_2) = x_1 x_2$ . If the appropriate uniform integrability holds, then one can pass expectations through (2.12) and (2.13), yielding

$$(2.14) \quad tE(s(t)-\sigma)^2 \rightarrow (4\sigma^2)^{-1}(EA_1^2 - 2\lambda EA_1Z_1 + \lambda^2 EZ_1^2)$$

$$(2.15) \quad tE(\bar{X}(t)-\mu)(s(t)-\sigma) \rightarrow (2\sigma)^{-1}(EA_1Z_1 - \lambda EZ_1^2) .$$

In other words, the covariance matrix  $C$  does legitimately correspond to the limiting covariances associated with  $\bar{X}(t)$  and  $s(t)$ . This is the content of our second application.

(2.16) THEOREM. If  $E(\tilde{Y}_0^8 + \tilde{Y}_1^8 + \tau_0^8 + \tau_1^8) < \infty$ , then (2.14) and (2.15) are valid.

Empirical evidence (see [11]) indicates that the asymptotic covariance  $(2\sigma)^{-1}(EA_1 Z_1 - \lambda E Z_1^2)$  is independent of the choice of the regeneration state. While the asymptotic variance of  $s(t)$  does depend on the regeneration state, the variance does not appear to be minimized by choosing the state with minimal expected inter-regeneration time. Both these issues will be studied theoretically in a future paper.

### 3. PROOF OF THEOREM 2.2

We will show that the collection of r.v.'s  $\Lambda = \{t^{p/2} |\bar{X}(t) - \mu|^p : t > 0\}$  is dominated by a uniformly integrable family, thereby showing that  $\Lambda$  is uniformly integrable.

Let  $N(t) = \max\{k \geq -1 : T(k) \leq t\}$ , and write

$$\int_0^t X_c(s) ds = \sum_{k=0}^{N(t)+2} Z_k - \int_t^{T(N(t)+2)} X_c(s) ds$$

where  $X_c(t) = X(t) - \mu$ . Then,

$$\begin{aligned} (3.1) \quad \left| \int_0^t X_c(s) ds \right|^p &\leq \left( \left| \sum_{k=1}^{N(t)+2} Z_k \right| + \left| Z_0 - \int_t^{T(N(t)+2)} X_c(s) ds \right| \right)^p \\ &\leq \left( \left| \sum_{k=1}^{N(t)+2} Z_k \right| + U_0 + U_{N(t)+1} + U_{N(t)+2} \right)^p \end{aligned}$$

where  $U_1 = \tilde{Y}_1 + |\mu|\tau_1$ . Observe that for non-negative numbers  $a_i (1 \leq i \leq k)$ ,  $|\sum_{i=1}^k a_i|^p \leq k \max_{1 \leq i \leq k} a_i^p = k^p \cdot \max_{1 \leq i \leq k} a_i^p \leq k^p \cdot \sum_{i=1}^k a_i^p$ , and hence it follows from (3.1) that

$$(3.2) \quad \tau^{p/2} |\bar{X}(\tau) - \mu|^p \leq 4^p \cdot \tau^{-p/2} \cdot \left( \left| \sum_{k=1}^{N(\tau)+2} z_k \right|^p + U_0^p + U_{N(\tau)+1}^p + U_{N(\tau)+2}^p \right) \\ \leq 4^p \cdot \tau^{-p/2} \cdot \left( \left| \sum_{k=1}^{N(\tau)+2} z_k \right|^p + \sum_{k=0}^{N(\tau)+2} U_k^p \right).$$

We will now prove that the right-hand side of (3.2) is uniformly integrable. First, observe that since  $E(\tilde{Y}_1^p + \tau_1^p) < \infty$ , it is immediate that  $EU_1^p < \infty$ , and so the strong law of large numbers applies, yielding

$$\frac{1}{n} \sum_{k=0}^{n-1} U_k^p \rightarrow EU_1^p \quad \text{a.s.}$$

Since  $N(\tau)/\tau \rightarrow (E\tau_1)^{-1}$  a.s., a simple argument proves that a.s.

$$\tau^{-p/2} \cdot \sum_{k=0}^{N(\tau)+2} U_k^p \rightarrow \begin{cases} EU_1^p / E\tau_1, & p = 2 \\ 0, & p > 2. \end{cases}$$

The continuous mapping principle and converging-together lemma (BILLINGSLEY (1968), pp. 25 and 31) prove that

$$(3.3) \quad \tau^{-p/2} \cdot \left( \left| \sum_{k=1}^{N(\tau)+2} z_k \right|^p + \sum_{k=0}^{N(\tau)+2} U_k^p \right) \Rightarrow \begin{cases} \sigma^p |N(0,1)|^p + EU_1^p / E\tau_1, & p = 2 \\ \sigma^p |N(0,1)|^p, & p > 2. \end{cases}$$

By Corollary 2 (11) of CHOW et al. (1979) (see also JANSON (1983), Theorem 2.3)

$$t^{-p/2} \cdot E\left(\left|\sum_{k=1}^{N(t)+2} z_k\right|^p\right) \rightarrow \sigma^p E|N(0,1)|^p.$$

On the other hand, Wald's identity applies to the sum in the  $U_k$ 's, yielding

$$E\left(\sum_{k=0}^{N(t)+2} U_k^p\right) = EU_0^p + E(N(t)+2)EU_1^p.$$

The elementary renewal theorem then gives

$$t^{-p/2} E\left(\sum_{k=0}^{N(t)+2} U_k^p\right) \rightarrow \begin{cases} EU_1^p/E\tau_1, & p = 2 \\ 0, & p > 2. \end{cases}$$

Consequently, the limit theorem (3.3) also holds in expectation. Thus, Theorem 5.4 of [2] shows that the right-hand side of (3.2) is uniformly integrable, proving the theorem.

#### 4. PROOF OF THEOREM 2.8

The proof has two parts; we first prove that (2.5) is valid under  $\hat{E}$  by appealing to Theorem 2.2, followed by showing that the obvious limit relation is valid on the right-hand side of (2.6).

Given that  $X$  is regenerative under  $P$ ,  $X$  is stationary regenerative under  $\hat{P}$ , where

$$(4.1) \quad \hat{P}\{X \in \cdot\} = \frac{1}{E\tau_1} E\left\{\int_{T(0)}^{T(1)} I(X \circ \theta_{T(0)+s} \in \cdot) ds\right\}$$

( $X \circ \theta_t$  is the stochastic process  $\{X(t+s): s \geq 0\}$ ). In order to apply Theorem 2.2, we need to show that  $\hat{E}(\tilde{Y}_0^2 + \tilde{Y}_1^2 + \tau_0^2 + \tau_1^2) < \infty$ . The

distribution of  $\mathbf{X}$  from the first cycle onwards is identical under both  $P$  and  $\hat{P}$  and hence

$$\hat{E}\tilde{Y}_1^2 = EY_1^2 < \infty$$

$$\hat{E}\tau_1^2 = E\tau_1^2 < \infty.$$

Some care is needed for the zero-th cycle, however. From (4.1), it follows that

$$\hat{P}\{\tilde{Y}_0 \in \cdot\} = \frac{1}{E\tau_1} E\left\{\int_{T(0)}^{T(1)} I\left(\int_{T(0)+s}^{T(1)} |X(u)| du \in \cdot\right) ds\right\}.$$

Hence

$$\hat{E}\tilde{Y}_0^2 = \frac{1}{E\tau_1} E\left\{\int_{T(0)}^{T(1)} \left(\int_{T(0)+s}^{T(1)} |X(u)| du\right)^2 ds\right\} \leq \frac{1}{E\tau_1} E\{\tau_1 \tilde{Y}_1^2\}.$$

By Hölder's inequality,  $E\tau_1 \tilde{Y}_1^2 \leq E^{1/3} \tau_1^3 \cdot E^{2/3} \tilde{Y}_1^3 < \infty$  by hypothesis. A similar analysis shows that

$$\hat{E}\tau_0^2 = \frac{1}{E\tau_1} E\left\{\int_{T(0)}^{T(1)} (T(1)-s)^2 ds\right\}$$

$$= \frac{E\tau_1^3}{3E\tau_1},$$

proving that  $E(\tilde{Y}_1^3 + \tau_1^3) < \infty$  suffices to guarantee that

$$t\hat{E}(\bar{X}(t)-\mu)^2 \rightarrow \sigma^2$$

as  $t \rightarrow \infty$ .

To prove the validity of taking formal limits in (2.6), it is enough to show that the integrand on the right-hand side of (2.6), namely

$$\left(1 - \frac{s}{t}\right) \hat{E}(X_c(0)X_c(s)) \cdot I(s \leq t)$$

is absolutely dominated by an integrable function. In other words, we are done if we prove that

$$(4.2) \quad \int_0^\infty |\hat{E}X_c(0)X_c(s)| ds < \infty.$$

Now, the left-hand side of (4.2) is itself dominated by

$$(4.3) \quad \int_0^\infty |\hat{E}X_c(0)X_c(s)I(T(0) > s)| ds + \int_0^\infty |\hat{E}X_c(0)X_c(s)I(T(0) \leq s)| ds.$$

The first integral in (4.3) can be bounded by

$$\int_0^\infty \hat{E}|X_c(0)X_c(s)|I(T(0) > s) ds = \hat{E} \int_0^{T(0)} |X_c(0)X_c(s)| ds.$$

By (4.1),

$$\begin{aligned} \hat{E} \int_0^{T(0)} |X_c(0)X_c(s)| ds &= \frac{1}{E\tau_1} E\left(\int_{T(0)}^{T(1)} |X_c(u)| \int_{T(0)+u}^{T(1)} |X_c(s)| ds du\right) \\ &\leq \frac{1}{E\tau_1} E\left(\int_{T(0)}^{T(1)} |X_c(u)| du\right)^2 \\ &\leq \frac{1}{E\tau_1} E(\tilde{Y}_1 + |u|\tau_1)^2 \end{aligned}$$

which is finite by hypothesis. We now turn to the second term in (4.3).

Observe that



$$\begin{aligned}
& \hat{E}X_c(0)X_c(s)I(T(0) \leq s) \\
&= \int_0^s \hat{E}\{X_c(0)X_c(s) | T(0) = u\} \hat{P}\{T(0) \in du\} \\
&= \int_0^s \hat{E}\{X_c(0) | T(0) = u\} EX_c(T(0)+s-u) \hat{P}\{T(0) \in du\} \\
&= \hat{E}X_c(0)a(s-T(0))I(T(0) \leq s),
\end{aligned}$$

where  $a(\cdot) = EX_c(T(0)+\cdot)$ . Hence,  $\hat{E}X_c(0)X_c(s)I(T(0) \leq s)$  is absolutely dominated by  $E|X_c(0)a(s-T(0))|I(T(0) \leq s)$ . By Fubini's theorem,

$$\begin{aligned}
& \int_0^\infty \hat{E}|X_c(0)a(s-T(0))| \cdot I(T(0) \leq s) ds \\
&= \hat{E}|X_c(0)| \cdot \int_{T(0)}^\infty |a(s-T(0))| ds \\
&= \hat{E}|X_c(0)| \cdot \beta,
\end{aligned}$$

where  $\beta = \int_0^\infty |a(u)| du$ . Since  $X$  is stationary under  $\hat{P}$ , it is evident that  $\hat{E}|X_c(0)| \leq E(\tilde{Y}_1 + |\mu|\tau_1)/E\tau_1$ . Hence, to complete the proof of the theorem, we need only show that  $\beta < \infty$  under our conditions.

Clearly, it is sufficient to prove that  $EX(T(0)+t) = \mu + O(t^{-2})$  under our hypotheses. Now,  $c(\cdot) = EX(T(0)+\cdot)$  satisfies the renewal equation  $c = b + c * F$  where  $F(dx) = P\{\tau_1 \in dx\}$  and  $b(t) = E\{X(T(0)+t) \cdot I(\tau_1 > t)\}$ . The solution of this renewal equation is given by  $c = b * U$ , where  $U = \sum_{k=0}^\infty F^{(k)}$  ( $F^{(k)}$  is the  $k^{\text{th}}$  convolution of  $F$ ). Consequently, the question of whether  $c(t) = \mu + O(t^{-2})$  is equivalent to showing that the rate of convergence of  $b * U$  to its limit  $\mu$  is  $O(t^{-2})$ .

This question has been extensively studied. If  $F$  has a density component, a sufficient condition for this relation is that (see NUMMELIN and TUOMINEN (1983), Theorem 4.2)

$$(4.4) \quad \int_0^{\infty} x^2 F(dx) < \infty .$$

A similar argument works for discrete-time. (To verify the conditions of [18], note that if a sequence is summable, the summands must converge to zero. The continuous analog is incorrect; this explains the stronger hypothesis in continuous time.)

#### 5. PROOF OF THEOREM 2.15

We first observe that  $\{t(\bar{X}(t)-\mu)^2: t > 0\}$  is uniformly integrable under our conditions. (Theorem 2.2.) Note that

$$t(\bar{X}(t)-\mu)(s(t)-\sigma) \leq (1/2)[t(\bar{X}(t)-\mu)^2 + t(s(t)-\sigma)^2] ,$$

so that the uniform integrability of  $\{t(\bar{X}(t)-\mu)(s(t)-\sigma): t > 0\}$  will follow immediately from uniform integrability of  $\{t(s(t)-\sigma)^2: t > 0\}$  (see p. 100 of [6]).

To obtain uniform integrability of  $\{t(s(t)-\sigma)^2: t > 0\}$ , we use the following lemma.

$$(5.1) \quad \text{LEMMA. For } x \geq 0, \sigma > 0, (x^{1/2}-\sigma)^2 \leq (4+\sigma^{-2}) \cdot (x-\sigma^2)^2.$$

PROOF. We look at two cases:  $x \leq \sigma^2/2$  and  $x > \sigma^2/2$ . If  $0 \leq x \leq \sigma^2/2$ , then  $|x^{1/2}-\sigma|^2 \leq |0-\sigma|^2 = 4(\sigma^2/4) \leq 4(x-\sigma^2)^2$ . On the other hand, if  $x > \sigma^2/2$ , then we use Taylor's formula to yield

$$|x^{1/2} - (\sigma^2)^{1/2}| = \frac{1}{2\xi^{1/2}} |x - \sigma^2|$$

where  $\xi$  lies between  $x$  and  $\sigma^2$ . Since  $x > \sigma^2/2$ , it follows that  $\xi^{-1/2} \leq 2^{1/2}/\sigma$  and hence

$$|x^{1/2} - \sigma|^2 \leq \frac{1}{\sigma} |x - \sigma^2|$$

for  $x > \sigma^2/2$ . The inequality follows immediately by combining the bounds for the two cases.

The lemma shows that  $t(s(t) - \sigma)^2 \leq (4 + 1/\sigma^2)t(v(t) - \sigma^2)^2$ . Our proof is therefore complete if we prove that  $\{t(v(t) - \sigma^2)^2: t > 0\}$  is uniformly integrable. We expand  $v(t) - \sigma^2$  as

$$\begin{aligned} v(t) - \sigma^2 &= \frac{1}{t} \sum_{k=1}^{N(t)} A_k + 2(\mu - \bar{X}(t)) \frac{1}{t} \sum_{k=1}^{N(t)} \tau_k Z_k \\ &\quad + (\mu - \bar{X}(t))^2 \cdot \frac{1}{t} \sum_{k=1}^{N(t)} \tau_k^2 \\ &\quad - \sigma^2(t - T(N(t)))/t. \end{aligned}$$

Thus, using the inequality  $(\sum_1^k |a_i|)^2 \leq k^2 \cdot \sum_1^k a_i^2$  (see Section 3) and  $|ab| \leq a^2 + b^2$ , one can bound  $t(v(t) - \sigma^2)^2$  by

$$\begin{aligned}
(5.2) \quad & t(v(t)-\sigma^2)^2 \leq 16 t^{-1} \left( \sum_{k=1}^{N(t)} A_k \right)^2 \\
& + 32 t^2 (\bar{X}(t) - \mu)^4 \\
& + 32 \left( \frac{1}{t} \sum_{k=1}^{N(t)} \tau_k Z_k \right)^4 \\
& + 16 t^2 (\bar{X}(t) - \mu)^8 \\
& + 16 \left( \frac{1}{t} \sum_{k=1}^{N(t)} \tau_k^2 \right)^4 \\
& + 16 \sigma^4 \sum_{k=1}^{N(t)+1} \tau_k^2 / t .
\end{aligned}$$

The second and fourth terms above are uniformly integrable, by Theorem 2.2. The uniform integrability of the sixth term is an easy consequence of Wald's identity. For the first term, observe that

$$(5.3) \quad t^{-1} \left( \sum_{k=1}^{N(t)} A_k \right)^2 \leq 9t^{-1} \left[ \left( \sum_{k=1}^{N(t)+2} A_k \right)^2 + \sum_{k=1}^{N(t)+2} A_k^2 \right] .$$

Uniform integrability of the first term on the right-hand side of (5.3) follows from Theorem 2 (ii) of [4], whereas uniform integrability of the second term is immediate from Wald's identity.

Similar arguments hold for the remaining terms in (5.2).

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